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The maximum load size effect for uncracked brittle structures

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The paper proceeds from the basis that the dominant source of the maximum load size effect for uncracked brittle structures is deterministic, and is related to the formation of a damage (fracture process) zone at a free surface. By modelling this damage behaviour in terms of the cohesive zone description, and by associating the maximum load with the attainment of an elastically calculated effective tensile failure stress, the paper projects the view that the effective stress is critically dependent on the applied loading induced stress gradient beneath the surface of a structure. The effective tensile failure stress increases with the steepness of the stress gradient, and we therefore have a ready explanation as to why the effective tensile failure stress for an uncracked bend beam increses as the beam depth decreases.

1. Introduction

It is a well known experimental fact that uncracked brittle structures or laboratory test specimens exhibit a maximum load size effect. This is best illustrated by considering the case of an uncracked bend beam specimen for which the maximum stress occurs at the tensile surface. Assuming elastic behaviour, the beam depth is d and the beam thickness is B, the bending moment M is equal to $Bd^2\sigma_s/6$ were σ_s is the tensile stress at the surface. Experiments, for example with unreinforced concrete beams, have shown that the maximum moment M_m that a beam is able to sustain is equal to $Bd^2\sigma_s/6$ with $\sigma_s = \sigma_m$ and where σ_m increases as the beam depth d decreases [1, 2]. The objective of this paper is to focus on the reason why σ_m should be geometry dependent, and more particularly as to why it should increase, in the case of a bend beam, as the beam depth d decreases.

As recently emphasized by Li and Baszant [3], the underlying cause of the size effect for brittle and quasi-brittle materials such as concrete, sea ice, rocks, tough ceramics and composites, stems from the fact that they exhibit damage (fracture process) zones which are able to grow in a stable manner prior to the attainment of maximum load. That being the case, the dominant source of the size effect is seemingly deterministic, and is related to the manner in which such a zone develops. The simplest way of quantifying this type of behaviour is to use the so-called cohesive zone description, whereby a single infinitesimally thin two dimensional cohesive zone starts to form at the surface when the tensile stress at the surface attains some critical value p_c. As the applied load (moment in the case of a bend beam) increases, the zone spreads away from the surface into the interior of the structure. The zone can be characterized by a material specific relation between the tensile stress (p) and the relative displacement (v) between the zone faces, with the maximum value of p, i.e., p_c, being at the leading edge of the cohesive zone. The zone is said to be fully developed when the stress falls to zero at the trailing edge of the zone, i.e., at the surface of the structure, a situation that is attained when the displacement v attains a critical value v_c, i.e., when the damage at the surface is sufficiently pronounced. The elastic "driving" stress, (calculated by assuming there is no zone), forcing the zone to extend into the solid away from the surface, depends on the distance from the surface (there is a linear stress fall-off with a bend beam), and this paper's prime objective is to show how the maximum load condition depends on the stress gradient. For a general p–v cohesive zone behaviour, the maximum load is attained prior to the cohesive zone's full development. However, so as to simplify the considerations, it will be assumed that the stress p remains constant at the value p_c within the cohesive zone until the displacement v attains the critical value v_c when the stress p falls abruptly to zero, this being the classic Dugdale–Bilby–Cottrell–Swinden (DBCS) representation [4, 5]. With this specific cohesive zone behaviour, the attainment of maximum load is associated with the full development of the cohesive zone. The results are expressed in terms of the way in which the maximum tensile stress σ_m at the surface, calculated assuming elastic behaviour and with no cohesive zone, depends on geometry, and in particular the stress gradient below the surface, and the cohesive zone properties.

2. The development of a cohesive zone from a planar surface in a semi-infinite solid

Consider the situation where there is a cohesive zone, within which the tensile stress is p_c, emanating from

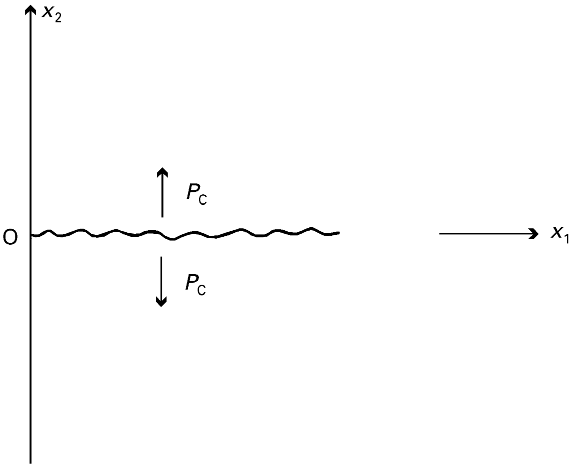

Figure 1 The model of a cohesive zone emanating from a planar surface in a semi-infinite solid.

the planar surface of a semi-infinite solid (Fig. 1). It is assumed that the tensile "driving" stress p_{22} along the plane $X_2 = 0$ in the absence of the cohesive zone is:

$$\sigma(x) = \sigma_L\left(1 - \frac{x}{h}\right) \tag{1}$$

where x is measured from $X_1 = 0$ along the X_1 axis; σ_L is the stress at the surface, and h is a length parameter which is a measure of the stress gradient. The isolated crack-infinite body solution can be used as an approximate solution for our situation by making a cut along the plane $X_1 = 0$, this procedure being exact for the analogous Mode III situation. If s is the length of the cohesive zone, standard elasticity theory gives the displacement between the faces of the cohesive zone at the position 0 (see Fig. 1), as [6]:

$$\Phi = \frac{4}{\pi E_0}\int_0^s \{\sigma(x) - p_c\} \ln\left\{\frac{s + (s^2 - x^2)^{1/2}}{s - (s^2 - x^2)^{1/2}}\right\} dx \tag{2}$$

where $E_0 = E/(1 - \nu^2)$ for plane strain deformation, E being Young's modulus and ν being Poisson's ratio. Furthermore since the stress p_{22} is finite ($= p_c$) at the in-board extremity $x = s$ of the cohesive zone, we must also satisfy the condition [6]:

$$\int_0^s \frac{\{\sigma(x) - p_c\}}{(s^2 - x^2)^{1/2}} dx = 0 \tag{3}$$

With $\sigma(x)$ being given by Equation 1, Equation 3 reduces to:

$$\frac{s}{h} = \frac{\pi(\sigma_L - p_c)}{2\sigma_L} \tag{4}$$

Furthermore, Equation 2, with $\Phi = v_c$, gives the criterion for the zone to be fully developed, i.e., for the attainment of maximum load, as:

$$v_c = \frac{4s(\sigma_L - p_c)}{E_0} - \frac{4\sigma_L}{\pi E_0 h}\int_0^s x \ln\left\{\frac{s + (s^2 - x^2)^{1/2}}{s - (s^2 - x^2)^{1/2}}\right\} dx \tag{5}$$

With $x = \sin\theta$, the integral I_1, as given by:

$$I_1 = \int_0^s x \ln\left\{\frac{s + (s^2 - x^2)^{1/2}}{s - (s^2 - x^2)^{1/2}}\right\} dx \tag{6}$$

reduces to:

$$I_1 = -2s^2\int_0^{\pi/2} \sin\theta\cos\theta \ln\left(\tan\frac{\theta}{2}\right) d\theta = s^2 \tag{7}$$

and consequently Equation 5 simplifies to:

$$v_c = \frac{4s(\sigma_L - p_c)}{E_0} - \frac{4\sigma_L s^2}{\pi E_0 h} \tag{8}$$

Equations 4 and 8 give the criterion for the attainment of maximum load, expressed in terms of the attainment of an elastically calculated stress σ_m at the surface, i.e., $\sigma_L = \sigma_m$ in Equations 4 and 8, as:

$$\frac{\sigma_m}{p_c} = 1 + \frac{E_0 v_c}{2\pi h p_c} + \left(\left(1 + \frac{E_0 v_c}{2\pi h p_c}\right)^2 - 1\right)^{1/2} \tag{9}$$

Thus with $G_F = v_c p_c$ being the specific energy associated with the failure within the cohesive zone and with $\chi = E_0 G_F/\pi p_c^2 h$, Equation 9 can be written in the form:

$$\frac{\sigma_m}{p_c} = 1 + \frac{\chi}{2} + \left(\chi + \frac{\chi^2}{4}\right)^{1/2} \tag{10}$$

or, with the terminology of Elices and Planas [7], whereby $l_{ch} \equiv E_0 G_F/p_c^2$ is a material size parameter,

$$\frac{\sigma_m}{p_c} = 1 + \frac{l_{ch}}{2\pi h} + \left(\frac{l_{ch}}{\pi h} + \frac{l_{ch}^2}{4\pi^2 h^2}\right)^{1/2} \tag{11}$$

The cohesive zone size s_m upon the attainment of maximum load is given by Equations 4, 10 and 11 as:

$$\frac{s_m}{h} = \frac{\pi}{2}\left(\left(\chi + \frac{\chi^2}{4}\right)^{1/2} - \frac{\chi}{2}\right) \tag{12}$$

or

$$\frac{s_m}{h} = \frac{\pi}{2}\left(\left(\frac{l_{ch}}{\pi h} + \frac{l_{ch}^2}{4\pi^2 h^2}\right)^{1/2} - \frac{l_{ch}}{2\pi h}\right) \tag{13}$$

The maximum stress results as given by Equations 10 and 11 are shown in Fig. 2, and the cohesive zone size results are shown in Fig. 3. The maximum stress results (Fig. 2) clearly highlight the fact that as the stress gradient increases, i.e., h decreases, then the maximum effective failure stress increases.

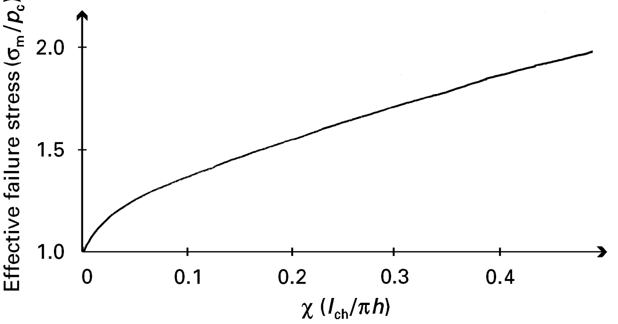

Figure 2 The maximum effective failure stress σ_m for the semi-infinite solid model expressed in terms of the parameter $\chi = E_0 G_F/\pi p_c^2 h = l_{ch}/\pi h$.

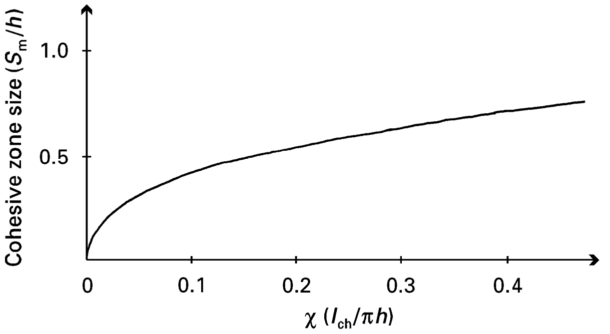

Figure 3 The cohesive zone size s_m upon the attainment of the maximum stress state for the semi-infinite solid model expressed in terms of the parameter $\chi = E_0 G_F / \pi p_c^2 h = l_{ch}/\pi h$.

3. The development of a cohesive zone from a planar surface in a finite depth bend specimen

Fig. 4 shows the model of a cohesive zone, within which the tensile stress is p_c, emanating from a planar surface of a bend specimen of finite depth d, thickness B and infinite length in the other direction. When the bending moment is M and the cohesive zone size is R, the relative displacement between the faces of the cohesive zone at the position O (see Fig. 4) is given by the following expression

$$\Phi = \frac{24MR}{E_0 B d^2} V(R/d) - \frac{4 p_c R}{E_0} V_1(R/d) \qquad (14)$$

the first term arising from the applied bending moment M, and the second term being due to the restraining stress within the cohesive zone; the functions V and V_1 have been given in graphical form [6]. Furthermore since the stress must be finite ($= p_c$) at the extremity of the cohesive zone, the stress intensity due to the applied moment at the in-board tip of the cohesive zone (viewed as a crack) must equate with the stress intensity due to the restraining stress within the cohesive zone, i.e.,

$$\frac{6M}{Bd^2} (\pi R)^{1/2} G(R/d) = p_c (\pi R)^{1/2} F(R/d) \qquad (15)$$

with the functions F and G having been given in graphical form [6]. Proceeding from the basis that the attainment of maximum load (moment) occurs when Φ attains the critical value v_c and that this state is associated with the attainment of an elastically calculated effective stress σ_m at the surface, i.e., $\sigma_m = 6M_m/Bd^2$ where M_m is the maximum moment, then Equations 14 and 15 allow σ_m to be obtained since they can be rewritten in the form:

$$\frac{\sigma_m}{p_c} = \frac{F(R/d)}{G(R/d)} \qquad (16)$$

and

$$\chi_* = \frac{E_0 G_F}{4 p_c^2 d} = \frac{l_{ch}}{4d} = \frac{R}{d}\left(\frac{F(R/d)V(R/d)}{G(R/d)} - V_1(R/\mathrm{d})\right) \qquad (17)$$

remembering that $G_F = p_c v_c$ and $l_{ch} = E_0 G_F / p_c^2$. It is therefore possible to obtain σ_m/p_c in terms of the parameter $\chi_* = E_0 G_F / 4 p_c^2 d = l_{ch}/4d$, by selecting values of R/w and using the tabulated values of the functions F, G, V and V_1. The results are shown in Table I and also in Fig. 5, while Fig. 6 shows the cohesive zone size R_m at maximum load. As with the semi-infinite solid model, the maximum stress results

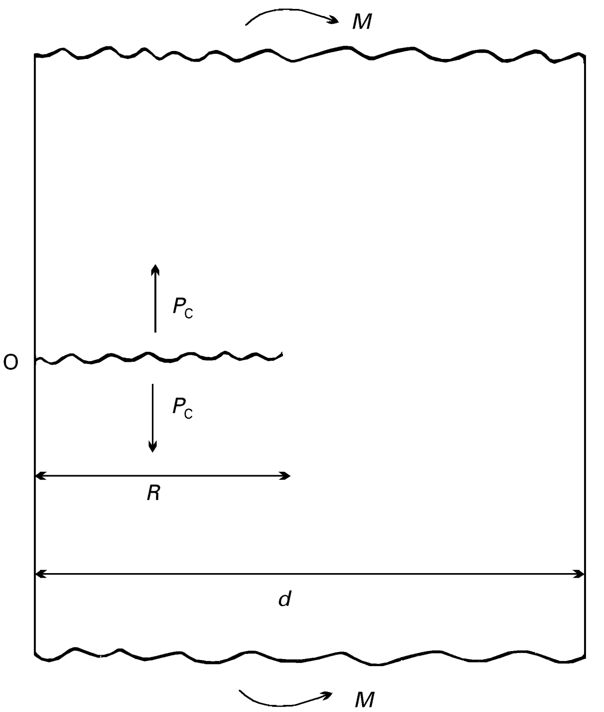

Figure 4 The model of a cohesive zone emanating from a planar surface in a finite depth bend specimen.

TABLE I σ_m for the finite depth bend specimen in terms of the parameter $\chi_* = E_0 G_F / 4 p_c^2 d = l_{ch}/4d$

R/d	$F(R/d)$	$G(R/d)$	$V(R/d)$	$V_1(R/d)$	σ_m/p_c	χ_*
0	1.122	1.122	1.460	1.460	1.000	0
0.1	1.206	1.042	1.457	1.543	1.157	0.014
0.2	1.369	1.047	1.602	1.828	1.308	0.053
0.3	1.655	1.109	1.898	2.327	1.492	0.152
0.4	2.107	1.247	2.389	3.278	1.690	0.303
0.5	2.825	1.497	3.200	5.000	1.887	0.519

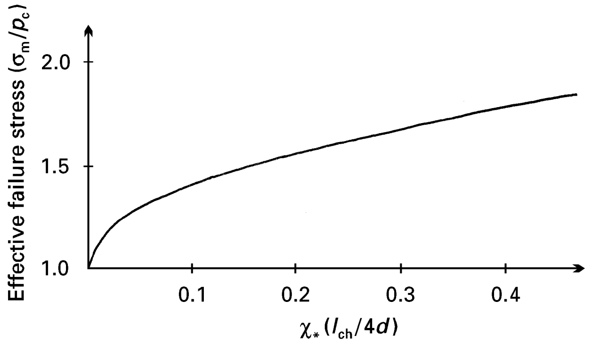

Figure 5 The maximum effective failure stress σ_m for the finite depth (d) bend specimen expressed in terms of the parameter $\chi_* = E_0 G_F / 4 p_c^2 d = l_{ch}/4d$.

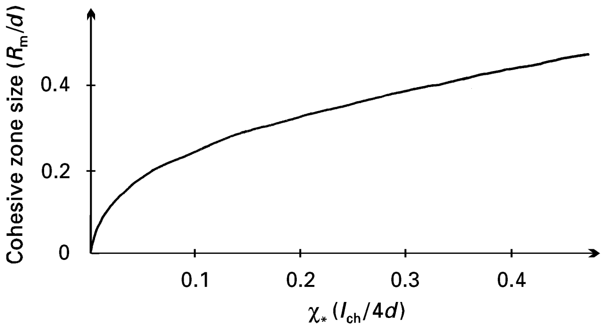

Figure 6 The cohesive zone size R_m upon the attainment of the maximum load state for the finite depth (d) bend specimen expressed in terms of the parameter $\chi_* = E_0 G_F / 4p_c^2 d = l_{ch}/4d$.

(Fig. 5) for the finite depth beam model highlight the fact that as the stress gradient increases, i.e., the beam depth d decreases, then the maximum effective stress increases.

4. Discussion

This paper has been concerned with the maximum load size effect for uncracked brittle structures. Proceeding from the basis that the dominant source of the size effect is deterministic, and is related to the development of a damage (fracture process) zone from a free surface, the paper has modelled the behaviour of such a zone by allowing all the non-linearity of material behaviour to be concentrated within an infinitesimally thin cohesive zone within which the restraining stress retains a constant value p_c until the relative displacement across the zone attains a critical value v_c when the restraining stress p falls abruptly from p_c to zero. The results have been expressed in terms of an elastically calculated effective tensile stress σ_m at the free surface of a structure.

Analysis of (a) the model of a semi-infinite solid subjected to a "driving" stress gradient (section 2) and (b) the model of a finite depth bend beam specimen (section 3) supports the view that σ_m is critically dependent on the stress gradient below the surface of a structure, with σ_m increasing the steeper is the stress gradient. The theoretical predictions are consistent with experimental results [1, 2] for uncracked bend beams, which show that σ_m increases as the beam depth decreases.

Before concluding this paper, it should be noted that the author, unlike Bazant [8], and Elices and Planas [7], has not assumed the existence of a pre-existing crack and then considered the smooth surface situation as a limiting case. The author prefers the approach that has been adopted in this paper, because it avoids the complexities associated with proceeding to a limit, and thereby allows the origins of the size effect to be projected in a more transparent manner.

5. Conclusion

This paper has shown how the maximum load size effect for uncracked brittle structures is related to the way in which the effective tensile stress at the surface is dependent on the applied stress gradient below the surface, in that the effective tensile stress increases with the steepness of his gradient, for example as the depth of a bend specimen decreases.

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